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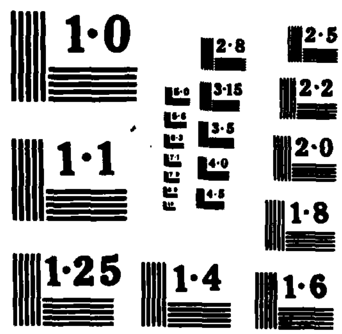
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In this paper the authors present a general model of the completion time of a single job on a computer system whose state changes according to a semi-Markov process. When the state of the system changes the job service is preempted. The job service is then resumed or restarted (with or without resampling) in the new state at, possibly, a different service rate. Different types of preemption disciplines are allowed in the model. Successive aggregation and transform techniques are used to obtain the Laplace Stieltjes Transform of the job completion time. The authors specialize to the case of Markovian state process. Finally the authors demonstrate the use of the techniques developed here by means of an application. --> cont keywords include:

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The Completion Time of a Job on Multi-Mode Systems*

V. G. Kulkarni†, V. F. Nicola‡ and K. S. Trivedi‡

Abstract

In this paper we present a general model of the completion time of a single job on a computer system whose state changes according to a semi-Markov process. When the state of the system changes the job service is preempted. The job service is then resumed or restarted (with or without resampling) in the new state at, possibly, a different service rate. Different types of preemption disciplines are allowed in the model. Successive aggregation and transform techniques are used to obtain the Laplace Stieltjes Transform of the job completion time. We specialize to the case of Markovian state process. Finally, we demonstrate the use of the techniques developed here by means of an application.

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1. *Introduction*

A computer system may change its state (e.g., its mode of operation) due to different events. The system behaviour can adequately be described by a stochastic process, referred to as the "structure-state process". Consider a job to be processed on such a system. The job service is preempted due to changes in the state of the system, and may be continued or repeated at, possibly, a different service rate. Clearly, the job completion time (i.e., the total time spent in the system to complete the job) is affected by changes in the system operation. It is important to distinguish different types of service-preemption interaction. In the preemptive-resume (*prs*) discipline, the job service is resumed from the point where it was preempted. In the preemptive-repeat-identical (*pri*) discipline, the job is restarted from its beginning. In the preemptive-repeat-different (*prd*) discipline, the job is restarted with a new work requirement which is statistically independent, and identically distributed to the original work requirement.

Several authors have studied the job completion time in special cases. Gaver [4] and Nicola [9] considered a single server system subject to different types of Poisson interruptions. In their system a job is serviced only in one state at a constant service rate. Castillo and Siewiorek [2] presented a model with two types of Poisson breakdowns and repair. Puri [10] studied the distribution of the cumulative service in the case where all preemptions are of the *prs* type; this is shown to be dual to the distribution of the completion time of a given job [7]. Under Markovian assumptions concerning the system changes (i.e., a Markovian structure-state process), we presented a detailed analysis in the cases where different, pure and mixed, types of preemptions are allowed in the model [6,7]. In [5], Iyer et al. considered the computational aspects and developed a procedure for the computation of the moments of the cumulative service in the case where all preemptions are of the *prs* type and the structure-state process is Markovian.

In this paper, we extend the results presented earlier to allow semi-Markovian system changes. Furthermore, all types of preemptions may be present in the same model. This is a general framework that includes all previous models as special cases, and extends the results to permit the analysis of fairly complex systems. In section 2, we describe the mathematical model. The job completion time is analyzed in section 3. We specialize to the case where the structure-state process is Markovian in section 4. In section 5, we demonstrate the use of the techniques developed here by means of an application.

2. The Basic Model

Consider a job with work requirement B, to be processed on a computer system. The work requirement is measured in work units, e.g., number of instructions to be executed. Assume that B is a positive random variable with cumulative distribution function $G(x) = P(B \le x)$. The computer system can be described by its structure-state process $\{Z(t), t \ge 0\}$, which is assumed to be a continuous time stochastic process defined on the state space $\{0,1,2,\ldots,n\}$. The work rate of the computer system is $r_i \ge 0$ (units of work per unit time) when it is in state i. The state 0 is an absorbing "failure" state with $r_0 = 0$. Each state in $\{1,2,\ldots,n\}$ is assumed to be *prs* or *pri* or *prd*, as defined in the previous section. Let $S_1(S_2,S_3)$ be the set of all *prs* (*pri*, *prd*) states. Thus S_1, S_2, S_3 is a partition of $\{1,2,\ldots,n\}$. We assume that the structure-state process is stochastically independent of the work requirement of the job.

Every time the system visits a *pri* or *prd* state all the previous work is lost and the job is restarted. Define $T(x)$ to be the amount of time needed to complete a job with work requirement of x units; and T to be the time needed to complete a job with random work requirement, B. We are interested in the distribution of T under the assumption that the job starts being processed immediately after the transition of the structure-state process to the initial state.

Define the distribution functions:

$$F_i(t,x) = P(T(x) \le t \mid Z(0)=i), \qquad t,x \ge 0,\ 1 \le i \le n$$

$$F(t,x) = P(T(x) \le t), \qquad t,x \ge 0$$

$$F_i(t) = P(T \le t \mid Z(0)=i), \qquad 1 \le i \le n,\ t \ge 0$$

$$F(t) = P(T \le t), \qquad t \ge 0.$$

Define the LSTs*:

$$F_i^{\sim}(s,x) = E(e^{-sT(x)} \mid Z(0)=i), \qquad x \ge 0, 1 \le i \le n,$$

$$F^{\sim}(s,x) = E(e^{-sT(x)}) = \sum_{i=1}^{n} F_i^{\sim}(s,x) P(Z(0)=i), \qquad x \ge 0$$

$$F_i^{\sim}(s) = E(e^{-sT} \mid Z(0)=i) = \int_0^{\infty} F_i^{\sim}(s,x)\, dG(x), \qquad 1 \le i \le n$$

* $(^{\sim})$ denotes the Laplace Stieltjes Transform (LST).

$$F^{*}(s) = E(e^{-sT}) = \sum_{i=1}^{n} F_i^{*}(s) P(Z(0)=i).$$

It is clear that $F_i^{*}(s,x)$ and $F_i^{*}(s)$ are the key quantities in the analysis of T. In the remaining analysis, we make the assumption that $\{Z(t), t \geq 0\}$ is a semi-Markov process (SMP). Let H be the holding (sojourn) time in the initial state, i.e.

$$H = \min\{t \geq 0 : Z(t) \neq Z(0)\}.$$

Let

$$Q_{ij}(x) = P\{H \leq x; Z(H+)=j \mid Z(0)=i\}, \quad 0 \leq i,j \leq n,$$

be the distribution of the sojourn time in state i, given that a transition to state j took place. $Q(x) = [Q_{ij}(x)]$ is called the kernel of the SMP. Let

$$Q_i(x) = \sum_{j=0}^{n} Q_{ij}(x), \quad 1 \leq i \leq n,$$

be the distribution of the sojourn time in state i, and define

$$Q_{ij}^{*}(s) = \int_0^{\infty} e^{-sx}\, dQ_{ij}(x)$$

and

$$Q_i^{*}(s) = \int_0^{\infty} e^{-sx}\, dQ_i(x).$$

3. *The Analysis of the Completion Time*

In this section we describe a systematic procedure to derive the distribution $F(t)$ of the job completion time T. The method of analysis is best described as "progressive aggregation". To begin with, we study the structure-state process restricted to move only in S_1, then we study it restricted to $S_1 \cup S_2$, and finally we study the general case, when it moves freely in $S_1 \cup S_2 \cup S_3$.

3.1. *The Analysis of the Completion Time in the pre Set of States*

Suppose the structure-state process is initially in S_1, and let

$$U_1 = \min\{t \geq 0 : Z(t) \notin S_1\}.$$

Thus, U_1 represents the first time the structure-state process visits a preemptive repeat state or the absorbing failure state. Now consider a job with work requirement x; its job completion time is $T(x)$. If $T(x) \leq U_1$ the job completes while the structure-state process is still in S_1, if $T(x) > U_1$ the job has to be either restarted in a state in $S_2 \cup S_3$ or the job is never completed due to an absorbing failure. Define

$$M_{1,i}^{\sim}(s,x) = E(e^{-sT(x)}; T(x) \leq U_1 \mid Z(0)=i), \quad i \in S_1,$$

to be the LST of the job completion time $T(x)$ when the job completion takes place before the structure-state process leaves the set S_1. Notice that $M_{1,i}^{\sim}(s,x)$ is the LST of a possibly defective random variable, since $P(T(x) \leq U_1 \mid Z(0)=i) \leq 1$.

Define, for $i \in S_1$ and $j \notin S_1$,

$$M_{1,i,j}^{\sim}(s,x) = E(e^{-sU_1}; T(x) > U_1, Z(U_1+)=j \mid Z(0)=i), \quad i \in S_1, \ j \notin S_1.$$

When the structure-state process starting in $i \in S_1$, leaves S_1 before the job completes and enters state $j \notin S_1$, the LST of the total time spent in the set S_1 is given by $M_{1,i,j}^{\sim}(s,x)$. Now let*

$$M_{1,i}^{\sim *}(s,w) = \int_0^\infty e^{-wx} M_{1,i}^{\sim}(s,x)\,dx, \quad i \in S_1$$

and

$$M_{1,i,j}^{\sim *}(s,w) = \int_0^\infty e^{-wx} M_{1,i,j}^{\sim}(s,x)\,dx, \quad i \in S_1, \ j \notin S_1.$$

The theorems below give the equations to obtain the above quantities.

Theorem 1. The double transforms $M_{1,i}^{\sim *}(s,w)$, $i \in S_1$, satisfy the following equations

$$M_{1,i}^{\sim *}(s,w) = \frac{r_i}{s+r_i w}\left[1-Q_i^{\sim}(s+r_i w)\right] + \sum_{k \in S_1} Q_{ik}^{\sim}(s+r_i w) M_{1,k}^{\sim *}(s,w), \quad i \in S_1. \tag{1}$$

Proof: Conditioning on H, the sojourn time in the initial state, we get

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* (*) denotes the Laplace transform.

$$E(e^{-sT(x)};T(x) \le U_1 \mid H=h, Z(0)=i) = \begin{cases} e^{-sx/r_i}, & \text{if } h \ge x/r_i \\ e^{-sh} \sum\limits_{k \in S_1} M^-_{1,k}(s, x-r_i h), & \text{if } h < x/r_i \end{cases} .$$

Unconditioning on H, we get

$$M^-_{1,i}(s,x) = \int_{x/r_i}^{\infty} e^{-sx/r_i} dQ_i(h) + \sum_{k \in S_1} \int_0^{x/r_i} e^{-sh} M^-_{1,k}(s, x-r_i h) dQ_{ik}(h)$$

$$= e^{-sx/r_i}(1-Q_i(x/r_i)) + \sum_{k \in S_1} \int_0^{x/r_i} e^{-sh} M^-_{1,k}(s, x-r_i h) dQ_{ik}(h).$$

Multiplying both sides by e^{-wx} and integrating, we get

$$M^{-*}_{1,i}(s,w) = \int_0^{\infty} e^{-wx} e^{-sx/r_i}(1-Q_i(x/r_i))dx + \sum_{k \in S_1} \int_0^{\infty} e^{-wx} \int_0^{x/r_i} e^{-sh} M^-_{1,k}(s, x-r_i h) dQ_{ik}(h) dx$$

which yields equations (1). *Q.E.D.*

Theorem 2. The double transforms $M^{-*}_{1,i,j}(s,w)$, $i \in S_1$, $j \notin S_1$, satisfy the following equations

$$M^{-*}_{1,i,j}(s,w) = \frac{1}{w} Q^-_{ij}(s+r_i w) + \sum_{k \in S_1} Q^-_{ik}(s+r_i w) M^{-*}_{1,k,j}(s,w), \quad i \in S_1, \; j \notin S_1. \tag{2}$$

Proof: Conditioning on H, the sojourn time in the initial state, we get

$$E(e^{-sU_1};T(x) > U_1, Z(U_1+) = j \mid H=h, Z(0)=i)$$

$$= \begin{cases} e^{-sh}, & \text{if } h < x/r_i \text{ and } Z(h)=j \notin S_1 \\ e^{-sh} M^-_{1,k,j}(s, x-r_i h), & \text{if } h < x/r_i \text{ and } Z(h)=k \in S_1 \end{cases} .$$

Unconditioning on H, we get

$$M^-_{1,i,j}(s,x) = \int_0^{x/r_i} e^{-sh} dQ_{ij}(h) + \sum_{k \in S_1} \int_0^{x/r_i} e^{-sh} M^-_{1,k,j}(s, x-r_i h) dQ_{ik}(h) .$$

Multiplying both sides by e^{-wx} and integrating, yield equations (2). *Q.E.D.*

Remark: If the structure-state process is a *CTMC*, the above theorems yield propositions 5.1 and 5.2 in [7].

Equations (1) and (2) can be solved to get $M_{1,i}^{-*}(s,w)$ and $M_{1,i,j}^{-*}(s,w)$, $i \in S_1$, $j \notin S_1$, which can be inverted with respect to w to obtain $M_{i,j}^{-}(s,x)$ and $M_{1,i,j}^{-}(s,x)$, respectively. If the structure-state process is a $CTMC$, this inversion can be carried out rather easily [7].

The quantities $M_{1,i}^{-}(s,x)$ and $M_{1,i,j}^{-}(s,x)$ completely describe the job completion process while the structure-state process is in S_1. Using these quantities we can analyze the job completion process while the structure-state process is in $S_{12}=S_1 \cup S_2$.

3.2. *The Analysis of the Completion Time in the prs-pri Sets of States*

Suppose the structure-state process is initially in S_{12} (i.e., in $S_1 \cup S_2$), and let

$$U_{12} = \min \{t \geq 0 : Z(t) \notin S_{12}\}.$$

Thus, U_{12} represents the time until the structure-state process visits a *prd* state or the absorbing failure state. Now consider a job with work requirement x; its completion time is $T(x)$. If $T(x) \leq U_{12}$, the job completes while the structure-state process is in S_{12}. If $T(x) > U_{12}$, the job has to be either restarted in a state in S_3 with a resampled work requirement or the job is never completed due to an absorbing failure. Define

$$M_{12,i}^{-}(s,x) = E(e^{-sT(x)}; T(x) \leq U_{12} \mid Z(0)=i), \quad i \in S_{12},$$

to be the LST of the job completion time $T(x)$ when the job completion takes place before the process leaves S_{12}. For $i \in S_{12}$ and $j \notin S_{12}$, define

$$M_{12,i,j}^{-}(s,x) = E(e^{-sU_{12}}; T(x) > U_{12}, Z(U_{12}+)=j \mid Z(0)=i), \quad i \in S_{12}, \; j \notin S_{12}.$$

When the structure-state process, starting in $i \in S_{12}$, leaves S_{12} before the job completes, and enters a state $j \notin S_{12}$, the LST of the total time spent in the set S_{12} is given by $M_{12,i,j}^{-}(s,x)$. Now let

$$g_i'(s,x) = e^{-sx/r_i}(1-Q_i(x/r_i)) + \sum_{k \in S_1} \int_0^{x/r_i} e^{-sh} M_{1,k}^{-}(s, x-r_i h)\, dQ_{ik}(h), \quad i \in S_2$$

and

$$h_{ij}'(s,x) = \int_0^{x/r_i} e^{-sh}\, dQ_{ij}(h) + \sum_{k \in S_1} \int_0^{x/r_i} e^{-sh} M_{1,k,j}^{-}(s, x-r_i h)\, dQ_{ik}(h), \quad i,j \in S_2.$$

The next theorem provides a method of computing the LSTs $M_{12,i}^{\cdot}(s,x)$, $i \in S_{12}$.

Theorem 3.

a) The conditional LSTs $M_{12,i}^{\cdot}(s,x)$, $i \in S_2$, satisfy the following equations

$$M_{12,i}^{\cdot}(s,x) = g_i'(s,x) + \sum_{j \in S_2} h_{ij}'(s,x) M_{12,j}^{\cdot}(s,x), \quad i \in S_2. \tag{3}$$

b) The conditional LSTs $M_{12,i}^{\cdot}(s,x)$, $i \in S_1$, are given by

$$M_{12,i}^{\cdot}(s,x) = M_{1,i}^{\cdot}(s,x) + \sum_{j \in S_2} M_{1,i,j}^{\cdot}(s,x) M_{12,j}^{\cdot}(s,x), \quad i \in S_1. \tag{4}$$

Proof.

a) Let $i \in S_2$ be the initial state. Conditioning on the sojourn time, H, in the initial state, we get

$$E(e^{-sT(x)}; T(x) \le U_{12} \mid Z(0)=i, H=h)$$

$$= \begin{cases} e^{-sx/r_i}, & \text{if } h \ge x/r_i \\ e^{-sh} \sum_{k \in S_2} M_{12,k}^{\cdot}(s,x) & \\ \quad + e^{-sh} \sum_{k \in S_1} [M_{1,k}^{\cdot}(s, x - r_i h) + \sum_{j \in S_2} M_{1,k,j}^{\cdot}(s, x - r_i h) M_{12,j}^{\cdot}(s,x)], & \text{if } h < x/r_i . \end{cases}$$

Unconditioning on H and rearranging yields equations (3).

b) Let $i \in S_1$ be the initial state. Then

$$E(e^{-sT(x)}; T(x) \le U_{12} \mid Z(0) = i)$$

$$= E(e^{-sT(x)}; T(x) \le U_1 \mid Z(0) = i) + E(e^{-sT(x)}; U_1 < T(x) \le U_{12} \mid Z(0) = i).$$

It follows that

$$M_{12,i}^{\cdot}(s,x) = M_{1,i}^{\cdot}(s,x) + \sum_{j \in S_2} E(e^{-sU_1}; T(x) > U_1, Z(U_1+) = j \mid Z(0) = i)\, M_{12,j}^{\cdot}(s,x)$$

which yields equations (4). *Q.E.D.*

Remark: If the structure-state process is a $CTMC$, the above theorem produces theorems 5.1 and 5.2 in [7].

The next theorem provides a method of computing the LSTs $M^{-}_{12,i,j}(s,x)$, $i \in S_{12}$, $j \notin S_{12}$. First, define

$$a_{ij}(s,x) = \int_0^{x/r_i} e^{-sh}\, dQ_{ij}(h) + \sum_{k \in S_1} \int_0^{x/r_i} e^{-sh} M^{-}_{1,k,j}(s, x - r_i h)\, dQ_{ik}(h), \; i \in S_2, \quad j \notin S_{12}$$

and

$$b_{ik}(s,x) = \int_0^{x/r_i} e^{-sh}\, dQ_{ik}(h) + \sum_{m \in S_1} \int_0^{x/r_i} e^{-sh} M^{-}_{1,m,k}(s, x - r_{ik})\, dQ_{im}(h), \quad i,k \in S_2 .$$

Theorem 4.

a) The conditional LSTs $M^{-}_{12,i,j}(s,x)$, $i \in S_2$, $j \notin S_{12}$, satisfy the following equations

$$M^{-}_{12,i,j}(s,x) = a_{ij}(s,x) + \sum_{k \in S_2} b_{ik}(s,x) M^{-}_{12,k,j}(s,x), \quad i \in S_2, \; j \notin S_{12} . \tag{5}$$

b) The conditional LSTs $M^{-}_{12,i,j}(s,x)$, $i \in S_1$, $j \notin S_{12}$, are given by

$$M^{-}_{12,i,j}(s,x) = M^{-}_{1,i,j}(s,x) + \sum_{k \in S_2} M^{-}_{1,i,k}(s,x) M^{-}_{12,k,j}(s,x), \quad i \in S_1, \; j \notin S_{12}. \tag{6}$$

Proof.

a) Let $i \in S_2$ be the initial state and $j \notin S_{12}$. Conditioning on H, the sojourn time in the initial state, we have

$$E(e^{-U_{12}}; T(x) > U_{12}, Z(U_{12}+) = j \mid H = h, Z(0) = i)$$

$$= \begin{cases} e^{-sh}, & \text{if } h < x/r_i \text{ and } Z(h) = j \notin S_{12} \\ e^{-sh} M^{-}_{12,k,j}(s,x), & \text{if } h < x/r_i \text{ and } Z(h) = k \in S_2 \\ e^{-sh} [M^{-}_{1,k,j}(s, x - r_i h) + \sum_{m \in S_2} M^{-}_{1,k,m}(s, x - r_i h) M^{-}_{12,m,j}(s,x)], & \text{if } h < x/r_i \text{ and } Z(h) = k \in S_1. \end{cases}$$

Unconditioning on H, yields

$$M^{-}_{12,i,j}(s,x) = \int_0^{x/r_i} e^{-sh}\, dQ_{ij}(h) + \sum_{k \in S_2} \int_0^{x/r_i} e^{-sh} M^{-}_{12,k,j}(s,x)\, dQ_{ik}(h)$$

$$+ \sum_{k \in S_1} [\int_0^{x/r_i} e^{-sh} M^{-}_{1,k,j}(s, x - r_i h)\, dQ_{ik}(h) + \sum_{m \in S_2} e^{-sh} M^{-}_{1,k,m}(s, x - r_i h) M^{-}_{12,m,j}(s,x)\, dQ_{ik}(h)].$$

Rearranging yields equation (5).

b) Let $i \in S_1$ be the initial state and $j \notin S_{12}$. Then

$$E(e^{-sU_{12}};T(x)>U_{12},Z(U_{12}+)=j \mid Z(0)=i)$$

$$= E(e^{-sU_1};T(x)>U_1,Z(U_1+)=j \mid Z(0)=i)$$

$$+ \sum_{k \in S_2} E(e^{-sU_1};T(x)>U_1,Z(U_1+)=k \mid Z(0)=i)\, E(e^{-sU_{12}};T(x)>U_{12},Z(U_{12}+)=j \mid Z(0)=k).$$

Rewriting yields equation (6). *Q.E.D.*

The stochastic properties of the job completion process while the structure-state process is in S_{12} are completely described by the quantities $M_{12,i}^{\sim}(s,x)$ and $M_{12,i,j}^{\sim}(s,x)$, $i \in S_{12}$, $j \notin S_{12}$. Knowing these quantities we can determine the distribution of T, the job completion time, as will be shown in the next section.

3.3. *The Analysis of the Completion Time in the pre-pri-prd Sets of States*

In this section we describe the method of computing the LSTs $F_i^{\sim}(s)$, $i \in S_1 \cup S_2 \cup S_3$. The main result is given in theorem 5. First, let us make the following definitions. For $i \in S_3$

$$g_i(s) = \int_0^\infty e^{-sx/r_i}(1-Q_i(x/r_i))dG(x)$$

$$+ \sum_{k \in S_2} \int_0^\infty M_{12,k}^{\sim}(s,x)\,\Big(\int_0^{x/r_i} e^{-sh}\, dQ_{ik}(h)\Big)dG(x)$$

$$+ \sum_{k \in S_1} \int_0^\infty \int_0^{x/r_i} e^{-sh} M_{12,k}^{\sim}(s,x-r_i h)\,dQ_{ik}(h)\,dG(x).$$

For $i,j \in S_3$

$$h_{ij}(s) = \int_0^\infty \int_0^{x/r_i} e^{-sh}\, dQ_{ij}(h)\,dG(x)$$

$$+ \sum_{k \in S_2} \int_0^\infty M_{12,k,j}^{\sim}(s,x)\,\Big(\int_0^{x/r_i} e^{-sh}\, dQ_{ik}(h)\Big)dG(x)$$

$$+ \sum_{k \in S_1} \int_0^{\infty} \int_0^{x/r_i} e^{-sh} M^{\sim}_{12,k,j}(s, x - r_i h) dQ_{ik}(h) dG(x).$$

For $i \in S_1 \cup S_2$

$$M^{\sim}_{12,i}(s) = \int_0^{\infty} M^{\sim}_{12,i}(s, x) dG(x)$$

and

$$M^{\sim}_{12,i,j}(s) = \int_0^{\infty} M^{\sim}_{12,i,j}(s, x) dG(x), \quad j \notin S_1 \cup S_2.$$

Theorem 5.

a) The LSTs of job completion time, $F_i^{\sim}(s)$, for $i \in S_3$, satisfy the following equations

$$F_i^{\sim}(s) = g_i(s) + \sum_{j \in S_3} h_{ij}(s) F_j^{\sim}(s), \quad i \in S_3. \tag{7}$$

b) The LSTs of job completion time, $F_i^{\sim}(s)$, for $i \in S_1 \cup S_2$, are given by

$$F_i^{\sim}(s) = M^{\sim}_{12,i}(s) + \sum_{j \in S_3} M^{\sim}_{12,i,j}(s) F_j^{\sim}(s), \quad i \in S_1 \cup S_2. \tag{8}$$

Proof.

a) Let $i \in S_3$ be the initial state. Conditioning on H, the sojourn time in the initial state, and on the initial work requirement B, we have:

$$E(e^{-sT} \mid B = x, H = h, Z(0) = i)$$

$$= \begin{cases} e^{-sx/r_i}, & \text{if } h \geq x/r_i \\ e^{-sh} \sum_{k \in S_3} F_k^{\sim}(s) + e^{-sh} \sum_{k \in S_2} [M^{\sim}_{12,k}(s, x) + \sum_{j \in S_3} M^{\sim}_{12,k,j}(s, x) F_j^{\sim}(s)] & \end{cases}$$

$$+ e^{-sh} \sum_{k \in S_1} [M^{\sim}_{12,k}(s, x - r_i h) + \sum_{j \in S_3} M^{\sim}_{12,k,j}(s, x - r_i h) F_j^{\sim}(s)], \quad \text{if } h < x/r_i.$$

Unconditioning on H, yields

$$E(e^{-sT} \mid B = x, Z(0) = i)$$

$$= e^{-sx/r_i}(1 - Q_i(x/r_i))$$

$$+ \sum_{k \in S_3} \int_0^{x/r_i} e^{-sh} F_k^{\sim}(s) dQ_{ik}(h)$$

$$+ \sum_{k \in S_2} \int_0^{x/r_i} e^{-sh} [M_{12,k}^{\sim}(s,x) + \sum_{j \in S_3} M_{12,k,j}^{\sim}(s,x) F_j^{\sim}(s)] dQ_{ik}(h)$$

$$+ \sum_{k \in S_1} \int_0^{x/r_i} e^{-sh} [M_{12,k}^{\sim}(s,x-r_i h) + \sum_{j \in S_3} M_{12,k,j}^{\sim}(s,x-r_i h) F_j^{\sim}(s)] dQ_{ik}(h).$$

Unconditioning on the initial work requirement, B, and rearranging yield equation (7).

b) Let $i \in S_{12}$ be the initial state. Conditioning on the initial work requirement, B, we have

$$E(e^{-sT} \mid B=x, Z(0)=i)$$

$$= E(e^{-sT}; T \le U_{12} \mid B=x, Z(0)=i)$$

$$+ \sum_{j \in S_3} E(e^{-sU_{12}}; T > U_{12}; Z(U_{12}+)=j \mid B=x, Z(0)=i)\, E(e^{-sT} \mid Z(0)=j)$$

$$= M_{12,i}^{\sim}(s,x) + \sum_{j \in S_3} M_{12,i,j}^{\sim}(s,x)\, F_j^{\sim}(s).$$

Unconditioning on B, yields equation (8). *Q.E.D.*

Theorem 5 provides the *LST* of the job completion time given the initial state. We recall that the job starts being serviced upon the transition of the structure-state process to the initial state.

The procedure to compute the distribution of the job completion time can now be stated as follows:

1. Obtain $M_{1,i}^{\sim *}(s,w)$, $i \in S_1$ by solving equations (1).
2. Invert $M_{1,i}^{\sim *}(s,w)$ with respect to w to obtain $M_{1,i}^{\sim}(s,x)$, $i \in S_1$.
3. Obtain $M_{1,i,j}^{\sim *}(s,w)$, $i \in S_1$, $j \notin S_1$ by solving equations (2).
4. Invert $M_{1,i,j}^{\sim *}(s,w)$ with respect to w to obtain $M_{1,i,j}^{\sim}(s,x)$, $i \in S_1$, $j \notin S_1$.
5. Obtain $M_{12,i}^{\sim}(s,x)$, $i \in S_2$, by solving equations (3).
6. Compute $M_{12,i}^{\sim}(s,x)$, $i \in S_1$, by using equations (4).
7. Obtain $M_{12,i,j}^{\sim}(s,x)$, $i \in S_2$, $j \notin S_{12}$, by solving equations (5).

8. Compute $M^{\sim}_{12,i,j}(s,x)$, $i \in S_1$, $j \in S_{12}$, by using equations (6).

9. Obtain $F_i^{\sim}(s)$, $i \in S_3$, by solving equations (7).

10. Compute $F_i^{\sim}(s)$, $i \in S_1 \cup S_2$, by using equations (8).

11. Compute $F^{\sim}(s) = \sum_{i=1}^{n} F_i^{\sim}(s) P(Z(0)=i)$.

12. Invert $F^{\sim}(s)$ to obtain $F(t)=P(T \leq t)$.

It should be noted that for a general *SMP* the problem of inversion is quite difficult. In the next section we illustrate the procedure for a simple system with a structure-state process that is a *CTMC*. In section 5 we also consider an application with a semi-Markovian structure-state process.

4. *A Special Case: A Markovian Structure-State Process*

Consider a system with four structure states $\{0,1,2,3\}$. State "0" is an absorbing failure state. State "1" is *prs*, state "2" is *pri* and state "3" is *prd*. Thus, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$. The structure-state process is a *CTMC* with λ_{ij} being the rate of transition from state "i" to state "j" $(j \neq i)$. The reward rates in the states "1", "2" and "3" are r_1, r_2 and r_3, respectively. As state "0" is absorbing, we have $\lambda_{0j}=0$, for all j, and $r_0 = 0$. Let $\lambda_i = \sum_{\substack{j=0 \\ j \neq i}}^{3} \lambda_{ij}$ be the total transition rate out of state "i".

We follow the procedure stated in section 3 to solve this example.

Step 1 & 2. Obtain $M^{\sim *}_{1,1}(s,w)$ from equation (1).

$$M^{\sim *}_{1,1}(s,w) = \frac{r_1}{s+r_1 w}[1-Q_1^{\sim}(s+r_1 w)]$$

$$= \frac{r_1}{s+r_1 w}[1-\frac{\lambda_1}{s+r_1 w+\lambda_1}] = \frac{r_1}{s+r_1 w+\lambda_1}.$$

Inverting with respect to w, we get

$$M^{-}_{1,1}(s,x) = e^{-(s+\lambda_1)x/r_1} = e_1$$

where we have defined

$$e_i = e^{-(s+\lambda_i)x/r_i}, \quad i = 1,2,3.$$

Step 3 & 4. Obtain $M^{-*}_{1,1,2}(s,w)$ and $M^{-*}_{1,1,3}(s,w)$ from equation (2).

$$M^{-*}_{1,1,2}(s,w) = \frac{1}{w} Q^{-}_{12}(s+r_1w) = \frac{\lambda_{12}}{w(s+r_1w+\lambda_1)} \quad ,$$

$$M^{-*}_{1,1,3}(s,w) = \frac{1}{w} Q^{-}_{13}(s+r_1w) = \frac{\lambda_{13}}{w(s+r_1w+\lambda_1)} \quad .$$

Inverting with respect to w, we get

$$M^{-}_{1,1,2}(s,x) = \frac{\lambda_{12}}{(s+\lambda_1)}(1-e_1),$$

$$M^{-}_{1,1,3}(s,x) = \frac{\lambda_{13}}{(s+\lambda_1)}(1-e_1).$$

Steps 5 & 6. Obtain $M^{-}_{12,1}(s,x)$ by solving equation (3).

$$M^{-}_{12,2}(s,x) = g_2'(s,x) + h_{22}'(s,x) M^{-}_{12,2}(s,x)$$

$$= \frac{g_2'(s,x)}{[1-h_{22}'(s,x)]}.$$

Now,

$$g_2'(s,x) = e^{-sx/r_2}(1-Q_2(x/r_2)) + \int_0^{x/r_2} e^{-sh} M^{-}_{1,1}(s,x-r_2h)\,dQ_{21}(h)$$

$$= e^{-(s+\lambda_2)x/r_2} + \lambda_{21} \int_0^{x/r_2} e^{-(s+\lambda_2)h}\; e^{-(s+\lambda_1)(x-r_2h)/r_1} dh$$

$$= e_2 + \lambda_{21} r_1 (e_1 - e_2)/r$$

where

$$r = r_1(s+\lambda_2) - r_2(s+\lambda_1)$$

and

$$h_{22}'(s,x) = \int_0^{x/r_2} e^{-sh} M_{1,1,2}^{-}(s,x-r_2h)\,dQ_{21}(h)$$

$$= \frac{\lambda_{12}\lambda_{21}}{(s+\lambda_1)} \int_0^{x/r_2} e^{-(s+\lambda_2)h}\ [1-e^{-(s+\lambda_1)(x-r_2h)/r_1}]\,dh$$

$$= \frac{\lambda_{12}\lambda_{21}}{(s+\lambda_1)(s+\lambda_2)}\ [1-\frac{1}{r}(r_1(s+\lambda_2)e_1-r_2(s+\lambda_1)e_2)].$$

Thus,

$$[1-h_{22}'(s,x)] = \frac{[(s+\lambda_1)(s+\lambda_2)-\lambda_{12}\lambda_{21}]+\lambda_{12}\lambda_{21}[\frac{1}{r}(r_1(s+\lambda_2)e_1-r_2(s+\lambda_1)e_2)]}{(s+\lambda_1)(s+\lambda_2)}$$

and

$$M_{12,2}^{-}(s,x) = \frac{g_2'(s,x)}{[1-h_{22}'(s,x)]}$$

$$= \frac{[(r-\lambda_{21}r_1)e_2+\lambda_{21}r_1e_1](s+\lambda_1)(s+\lambda_2)}{r\,[(s+\lambda_1)(s+\lambda_2)-\lambda_{12}\lambda_{21}]+\lambda_{12}\lambda_{21}[r_1(s+\lambda_2)e_1-r_2(s+\lambda_1)e_2]}.$$

From equation (4), we have

$$M_{12,1}^{-}(s,x) = M_{1,1}^{-}(s,x) + M_{1,1,2}^{-}(s,x)M_{12,2}^{-}(s,x)$$

$$= e_1 + \frac{\lambda_{12}}{(s+\lambda_1)}\ (1-e_1)M_{12,2}^{-}(s,x).$$

Steps 7 & 8. Obtain $M_{12,2,3}^{-}(s,x)$ by solving equation (5).

$$M_{12,2,3}^{-}(s,x) = a_{23}(s,x) + b_{22}(s,x)M_{12,2,3}^{-}(s,x)$$

$$= \frac{a_{23}(s,x)}{[1-b_{22}(s,x)]}.$$

Now,

$$a_{23}(s,x) = \int_0^{x/r_2} e^{-sh}\,dQ_{23}(h) + \int_0^{x/r_2} e^{-sh} M_{1,1,3}^{-}(s,x-r_2h)\,dQ_{21}(h)$$

$$= \lambda_{23}\int_0^{x/r_2} e^{-(s+\lambda_2)h}\,dh\ +\ \frac{\lambda_{21}\lambda_{13}}{(s+\lambda_1)}\int_0^{x/r_2} e^{-(s+\lambda_2)h}\ [1-e^{-(s+\lambda_1)(x-r_2h)/r_1}]\,dh$$

$$= \frac{[\lambda_{23}(s+\lambda_1)+\lambda_{21}\lambda_{13}](1-e_2)-\lambda_{21}\lambda_{13}r_1(s+\lambda_2)(e_1-e_2)/r}{(s+\lambda_1)(s+\lambda_2)} .$$

$$b_{22}(s,x) = \int_0^{x/r_2} e^{-sh} M^*_{1,1,2}(s, x-r_2h)\, dQ_{21}(h)$$

$$= \frac{\lambda_{12}\lambda_{21}}{(s+\lambda_1)} \int_0^{x/r_2} e^{-(s+\lambda_2)h} \left[1-e^{-(s+\lambda_1)(x-r_2h)/r_1}\right] dh$$

$$= \frac{\lambda_{12}\lambda_{21}[r+r_2(s+\lambda_1)e_2-r_1(s+\lambda_2)e_1]}{r(s+\lambda_1)(s+\lambda_2)} .$$

Thus,

$$[1-b_{22}(s,x)] = \frac{r[(s+\lambda_1)(s+\lambda_2)-\lambda_{12}\lambda_{21}]-\lambda_{12}\lambda_{21}[r_2(s+\lambda_1)e_2-r_1(s+\lambda_2)e_1]}{r(s+\lambda_1)(s+\lambda_2)}$$

and

$$M^*_{12,2,3}(s,x) = \frac{a_{23}(s,x)}{[1-b_{22}(s,x)]}$$

$$= \frac{r[\lambda_{23}(s+\lambda_1)(1-e_2)+\lambda_{21}\lambda_{13}]+\lambda_{21}\lambda_{13}[r_2(s+\lambda_1)e_2-r_1(s+\lambda_2)e_1]}{r[(s+\lambda_1)(s+\lambda_2)-\lambda_{12}\lambda_{21}]+\lambda_{12}\lambda_{21}[r_1(s+\lambda_2)e_1-r_2(s+\lambda_1)e_2]} .$$

From equation (6), we get

$$M^*_{12,1,3}(s,x) = M^*_{1,1,3}(s,x) + M^*_{1,1,2}(s,x) M^*_{12,2,3}(s,x)$$

$$= \frac{(1-e_1)}{(s+\lambda_1)} [\lambda_{13}+\lambda_{12}M^*_{12,2,3}(s,x)].$$

Steps 9 & 10. Obtain $F^*_3(s)$ by solving equation (7).

$$F^*_3(s) = g_3(s)+h_{33}(s)F^*_3(s)$$

$$= \frac{g_3(s)}{[1-h_{33}(s)]} .$$

Now,

$$g_3(s) = \int_0^\infty e^{-sx/r_3}(1-Q_3(x/r_3))\, dG(x)$$

$$+ \int_0^\infty \int_0^{x/r_3} e^{-sh} M^*_{12,2}(s,x)\, dQ_{32}(h)\, dG(x)$$

$$+\int_0^\infty \int_0^{x/r_3} e^{-sh} M^{*}_{12,1}(s, x-r_3h)\, dQ_{31}(h)\, dG(x)$$

$$=\int_0^\infty e^{-(s+\lambda_3)x/r_3} dG(x)$$

$$+\lambda_{32}\int_0^\infty \int_0^{x/r_3} e^{-(s+\lambda_3)h} M^{*}_{12,2}(s, x)\, dh \;\; dG(x)$$

$$+\lambda_{31}\int_0^\infty \int_0^{x/r_3} e^{-(s+\lambda_3)h} M^{*}_{12,1}(s, x-r_3h)\, dh \;\; dG(x)$$

$$= e_3^{*} + \frac{\lambda_{32}}{(s+\lambda_3)}\int_0^\infty (1-e_3) M^{*}_{12,2}(s, x)\, dG(x) + \lambda_{31}\int_0^\infty \int_0^{x/r_3} e^{-(s+\lambda_3)h} M^{*}_{12,1}(s, x-r_3h)\, dh \;\; dG(x)$$

with

$$e_i^{*} = G^{*}((s+\lambda_i)x/r_i) = \int_0^\infty e^{-(s+\lambda_i)x/r_i} dG(x), \qquad i = 1,2,3$$

and

$$h_{33}(s) = \int_0^\infty \int_0^{x/r_3} e^{-sh} M^{*}_{12,2,3}(s, x)\, dQ_{32}(h)\, dG(x) + \int_0^\infty \int_0^{x/r_3} e^{-sh} M^{*}_{12,1,3}(s, x-r_3h)\, dQ_{31}(h)\, dG(x)$$

$$= \lambda_{32}\int_0^\infty \int_0^{x/r_3} e^{-(s+\lambda_3)h} M^{*}_{12,2,3}(s, x)\, dh \;\; dG(x) + \lambda_{31}\int_0^\infty \int_0^{x/r_3} e^{-(s+\lambda_3)h} M^{*}_{12,1,3}(s, x-r_3h)\, dh \;\; dG(x)$$

$$= \frac{\lambda_{32}}{(s+\lambda_3)}\int_0^\infty (1-e_3) M^{*}_{12,2,3}(s, x)\, dG(x) + \lambda_{31}\int_0^\infty \int_0^{x/r_3} e^{-(s+\lambda_3)h} M^{*}_{12,1,3}(s, x-r_3h)\, dh \;\; dG(x).$$

Thus,

$$F_3^{*}(s) = \frac{g_3(s)}{[1-h_{33}(s)]}$$

$$= \frac{(s+\lambda_3)e_3^{*} + \lambda_{32}\int_0^\infty (1-e_3) M^{*}_{12,2}(s, x)\, dG(x) + \lambda_{31}(s+\lambda_3))\int_0^\infty \int_0^{x/r_3} e^{-(s+\lambda_3)h} M^{*}_{12,1}(s, x-r_3h)\, dh \;\; dG(x)}{(s+\lambda_3) - \lambda_{32}\int_0^\infty (1-e_3) M^{*}_{12,2,3}(s, x)\, dG(x) - \lambda_{31}(s+\lambda_3)\int_0^\infty \int_0^{x/r_3} e^{-(s+\lambda_3)h} M^{*}_{12,1,3}(s, x-r_3h)\, dh \;\; dG(x)}.$$

From equations (8), we have

$$F_1^{*} = M^{*}_{12,1}(s) + M^{*}_{12,1,3}(s) F_3^{*}(s)$$

and

$$F_2^{*}(s) = M_{12,2}^{*}(s) + M_{12,2,3}^{*}(s) F_3^{*}(s).$$

Clearly,

$$F^{*}(s) = \sum_{i=1}^{3} P(Z(0)=i) F_i^{*}(s).$$

5. ***An Application: A System with Two Types of Breakdown and Repair***

Consider a job to be executed on a computer system. The system is subject to three types of Poisson failures:

1) Preemptive-resume (*prs*) failures, which occur only when the system is operating normally, i.e., in state "1", at a rate λ_{12}. Following a *prs* failure the system undergoes a breakdown period, of a general distribution $R_2(x)$ and *LST* $R_2^{*}(s)$, after which the interrupted job resumes service.

2) Preemptive-repeat-different (*prd*) failures, which occur when the system is either operating normally or undergoing a *prs* breakdown period, i.e., either in state "1" or in state "2", at rates λ_{13} and λ_{23}, respectively. Following a *prd* failure the system undergoes a breakdown period, of a general distribution $R_3(x)$ and *LST* $R_3^{*}(s)$, after which a new independent job with work requirement distribution $G(x)$ is restarted.

3) System failures, which occur in any state of the system operation, i.e., states "1", "2" and "3", at rates λ_{10}, λ_{20} and λ_{30}, respectively. These are absorbing failures. A state transition diagram representing the system is shown in figure 1. In this case we partition the states as follows (in accordance with section 3): the *prs* set, $S_1=\{1,2\}$, and the *prd* set, $S_3=\{3\}$. Note that S_2 is an empty set, since there are no *pri* states. Obviously, we set $r_1=1$ and $r_2=r_3=0$.

In order to obtain the *LST* of the job completion time, $F_1^{*}(s)$, we follow the procedure in section 3. Let H be the holding time in the initial state, then

$$Q_{1j}(x) = P(H \le x; Z(H+) = j \mid Z(0) = 1), \quad j=0,2,3$$

and

$$Q_1(x) = P(H \le x \mid Z(0) = 1) = Q_{10}(x) + Q_{12}(x) + Q_{13}(x).$$

Let $\lambda_1 = \lambda_{10} + \lambda_{12} + \lambda_{13}$, then from the Poisson property of the different types of failures, it follows that

$$Q_{1j}^{\sim}(s) = \frac{\lambda_{1j}}{s+\lambda_1} \ , \quad j=0,2,3$$

and

$$Q_1^{\sim}(s) = \frac{\lambda_1}{s+\lambda_1}.$$

Steps 1 & 2. Obtain $M_{1,1}^{\sim *}(s,w)$ and $M_{1,2}^{\sim *}(s,w)$ from equations (1).

$$M_{1,1}^{\sim *}(s+w) = \frac{1-Q_1^{\sim}(s+w)}{s+w} + Q_{12}^{\sim}(s+w)M_{1,2}^{\sim *}(s,w) \ ,$$

$$M_{1,2}^{\sim *}(s,w) = Q_{21}^{\sim}(s)M_{1,1}^{\sim *}(s,w).$$

It follows that

$$M_{1,1}^{\sim *}(s,w) = [s+w+\lambda_1-\lambda_{12}Q_{21}^{\sim}(s)]^{-1}.$$

Inverting with respect to w, we have

$$M_{1,1}^{\sim}(s,x) = e^{-[s+\lambda_1-\lambda_{12}Q_{21}^{\sim}(s)]x}$$

and

$$M_{1,2}^{\sim}(s,x) = Q_{21}^{\sim}(s)M_{1,1}^{\sim}(s,x) = Q_{21}^{\sim}(s)e^{-[s+\lambda_1-\lambda_{12}Q_{21}^{\sim}(s)]x} .$$

Steps 3 & 4. Obtain $M_{1,1,3}^{\sim *}(s,w)$ and $M_{1,2,3}^{\sim *}(s,w)$ from equations (2).

$$M_{1,1,3}^{\sim *}(s,w) = \frac{Q_{13}^{\sim}(s+w)}{w} + Q_{12}^{\sim}(s+w)M_{1,2,3}^{\sim *}(s,w),$$

$$M_{1,2,3}^{\sim *}(s,w) = \frac{Q_{23}^{\sim}(s)}{w} + Q_{21}^{\sim}(s)M_{1,1,3}^{\sim *}(s,w).$$

It follows that

$$M_{1,1,3}^{\sim *}(s,w) = \frac{\lambda_{13}+\lambda_{12}Q_{23}^{\sim}(s)}{w[s+w+\lambda_1-\lambda_{12}Q_{21}^{\sim}(s)]} .$$

Inverting with respect to w, we get

$$M^{\tilde{}}_{1,1,3}(s,x) = \frac{\lambda_{13}+\lambda_{12}Q^{\tilde{}}_{23}(s)}{s+\lambda_1-\lambda_{12}Q^{\tilde{}}_{21}(s)}\left(1-e^{-[s+\lambda_1-\lambda_{12}Q^{\tilde{}}_{21}(s)]x}\right)$$

and

$$M^{\tilde{}}_{1,2,3}(s,x) = Q^{\tilde{}}_{23}(s) + Q^{\tilde{}}_{21}(s)M^{\tilde{}}_{1,1,3}(s,x)$$

$$= \frac{(s+\lambda_1)Q^{\tilde{}}_{23}(s) + \lambda_{13}Q^{\tilde{}}_{21}(s) - Q^{\tilde{}}_{21}(s)[\lambda_{13}+\lambda_{12}Q^{\tilde{}}_{23}(s)]e^{-[s+\lambda_1-\lambda_{12}Q^{\tilde{}}_{21}(s)]x}}{s+\lambda_1-\lambda_{12}Q^{\tilde{}}_{21}(s)}.$$

Steps 5 & 6 & 7 & 8. Since S_2 is an empty set, it is clear that

$$M^{\tilde{}}_{12,1}(s,x) = M^{\tilde{}}_{1,1}(s,x),$$

$$M^{\tilde{}}_{12,2}(s,x) = M^{\tilde{}}_{1,2}(s,x),$$

$$M^{\tilde{}}_{12,1,3}(s,x) = M^{\tilde{}}_{1,1,3}(s,x),$$

$$M^{\tilde{}}_{12,2,3}(s,x) = M^{\tilde{}}_{1,2,3}(s,x).$$

Steps 9 & 10. Obtain $F^{\tilde{}}_3(s)$ from equation (7).

$$F^{\tilde{}}_3(s) = g_3(s) + h_{33}(s)F^{\tilde{}}_3(s),$$

where

$$g_3(s) = \int_0^\infty\int_0^\infty e^{-sh}M^{\tilde{}}_{1,1}(s,x)dQ_{31}(h)dG(x)$$

$$= M^{\tilde{}}_{1,1}(s)Q^{\tilde{}}_{31}(s)$$

and

$$h_{33}(s) = \int_0^\infty\int_0^\infty e^{-sh}\, M^{\tilde{}}_{1,1,3}(s,x)dQ_{31}(h)dG(x)$$

$$= M^{\tilde{}}_{1,1,3}(s)Q^{\tilde{}}_{31}(s),$$

with

$$M^{\sim}_{1,1}(s) = \int_0^\infty M^{\sim}_{1,1}(s,x)dG(x)$$

$$= G^{\sim}(s+\lambda_1-\lambda_{12}Q^{\sim}_{21}(s))$$

and

$$M^{\sim}_{1,1,3}(s) = \int_0^\infty M^{\sim}_{1,1,3}(s,x)dG(x)$$

$$= \frac{\lambda_{13}+\lambda_{12}Q^{\sim}_{23}(s)}{s+\lambda_1-\lambda_{12}Q^{\sim}_{21}(s)}\left[1-G^{\sim}(s+\lambda_1-\lambda_{12}Q^{\sim}_{21}(s))\right].$$

It follows that

$$F^{\sim}_3(s) = \frac{g_3(s)}{1-h_{33}(s)}$$

$$= \frac{Q_{31}(s)M^{\sim}_{1,1}(s)}{1-Q^{\sim}_{31}(s)M^{\sim}_{1,1,3}(s)}.$$

From equation (8), we have

$$F^{\sim}_1(s) = M^{\sim}_{1,1}(s) + M^{\sim}_{1,1,3}(s)F^{\sim}_3(s)$$

$$= \frac{M^{\sim}_{1,1}(s)}{1-Q^{\sim}_{31}(s)M^{\sim}_{1,1,3}(s)}$$

$$= \frac{G^{\sim}(s+\lambda_1-\lambda_{12}Q^{\sim}_{21}(s))}{1-\dfrac{Q^{\sim}_{31}(s)(\lambda_{13}+\lambda_{12}Q^{\sim}_{23}(s))}{s+\lambda_1-\lambda_{12}Q^{\sim}_{21}(s)[1-G^{\sim}(s+\lambda_1-\lambda_{12}Q^{\sim}_{21}(s))]}}.$$

Now, let us determine the LSTs $Q^{\sim}_{31}(s)$, $Q^{\sim}_{21}(s)$ and $Q^{\sim}_{23}(s)$. We have

$$Q_{31}(x) = P(H \le x; Z(H+) = 1 \mid Z(0) = 3)$$

$$= \int_0^x e^{-\lambda_{30}t}\, dR_3(t).$$

Thus

$$Q^{\sim}_{31}(s) = \int_0^\infty e^{-sx}\, dQ_{31}(x)$$

/

$$= \int_0^\infty e^{-sx} \; e^{-\lambda_{30}x} \, dR_3(x) = R_3^*\,(s+\lambda_{30}).$$

We also have

$$Q_{2j}(x) = P(H \le x; Z(H+) = j \mid Z(0) = 2), \quad j = 0,1,3.$$

Then

$$Q_{21}(x) = \int_0^x e^{-(\lambda_{20}+\lambda_{23})t} \, dR_2(t)$$

and

$$Q_{21}^*\,(s) = \int_0^\infty e^{-sx} \, dQ_{21}(x)$$

$$= \int_0^\infty e^{-sx} \; e^{-(\lambda_{20}+\lambda_{23})x} \, dR_2(x) = R_2^*\,(s+\lambda_{20}+\lambda_{23}).$$

Similarly,

$$Q_{23}^*\,(x) = \int_0^x \lambda_{23} \, e^{-\lambda_{23}t} \; e^{-\lambda_{20}t} \, [1-R_2(t)] \, dt$$

and

$$Q_{23}^*\,(s) = \int_0^\infty e^{-sx} \; dQ_{23}(x)$$

$$= \lambda_{23} \int_0^\infty e^{-sx} \, e^{-(\lambda_{23}+\lambda_{20})x} \, [1-R_2(x)] \, dx$$

$$= \frac{\lambda_{23}}{(s+\lambda_{20}+\lambda_{23})} \, [1-R_2^*\,(s+\lambda_{20}+\lambda_{23})].$$

Let $\lambda_2 = \lambda_{20} + \lambda_{23}$ and $\lambda_3 = \lambda_{30}$. Then substituting in $F_1^*\,(s)$, we finally get

$$F_1^*\,(s) = \frac{G^*\,(s+\lambda_1-\lambda_{12}R_2^*\,(s+\lambda_2))}{1-\dfrac{R_3^*\,(s+\lambda_3)[\lambda_{13}(s+\lambda_2)+\lambda_{12}\lambda_{23}(1-R_2^*\,(s+\lambda_2))][1-G^*\,(s+\lambda_1-\lambda_{12}R_2^*\,(s+\lambda_2))]}{(s+\lambda_2)[s+\lambda_1-\lambda_{12}R_2^*\,(s+\lambda_2)]}}.$$

In the case where system failures may not occur, i.e., $\lambda_{10} = \lambda_{20} = \lambda_{30} = 0$, and $\lambda_{13} = \lambda_{23}$, the above system corresponds to a model considered by Castillo and Siewiorek [2]. In this case $F_1^*\,(s)$ reduces to

$$F_1^{*}(s) = \frac{(s+\lambda_{13})G^{*}(s+\lambda_1-\lambda_{12}R_2^{*}(s+\lambda_{13}))}{(s+\lambda_{13})-\lambda_{13}R_3^{*}(s)[1-G^{*}(s+\lambda_1-\lambda_{12}R_2^{*}(s+\lambda_{13}))]}.$$

Furthermore, if the job service requirement is deterministic and equal to x (i.e., $G^{*}(s) = e^{-sx}$) then

$$F_1^{*}(s) = \frac{(s+\lambda_{13})e^{-[s+\lambda_1-\lambda_{12}R_2^{*}(s+\lambda_{13})]x}}{(s+\lambda_{13})-\lambda_{13}R_3^{*}(s)[1-e^{-[s+\lambda_1-\lambda_{12}R_2^{*}(s+\lambda_{13})]x}]}.$$

The mean completion time of a job, $E(T)$, is given by

$$E(T) = -F_1^{*\,\prime}(0) = [\frac{1}{\lambda_{13}} + E(R_3)]\,[e^{[\lambda_1-\lambda_{12}R_2^{*}(\lambda_{13})]x}-1]$$

where $E(R_3) = -R_3^{*\,\prime}(0)$, is the mean of the breakdown period after *prd* failures.

6. *Conclusions*

We have presented a general model for the analysis of job completion time on a system subject to changes in its structure due to different events. The system behaviour is described by a semi-Markov process. A change in the system operation preempts job service which may later be resumed or restarted (with or without resampling) at, possibly, a different service rate. We have derived a procedure to obtain the distribution of the job completion time. A closed form expression for the Laplace Stieltjes Transform of the job completion time is obtained in special cases. In the general case we resort to numerical techniques; this is an open problem for further research. In this paper we have restricted our attention to the execution of a single job on the system. The obvious extension to the case where an additional delay may be experienced due to queueing is being investigated.

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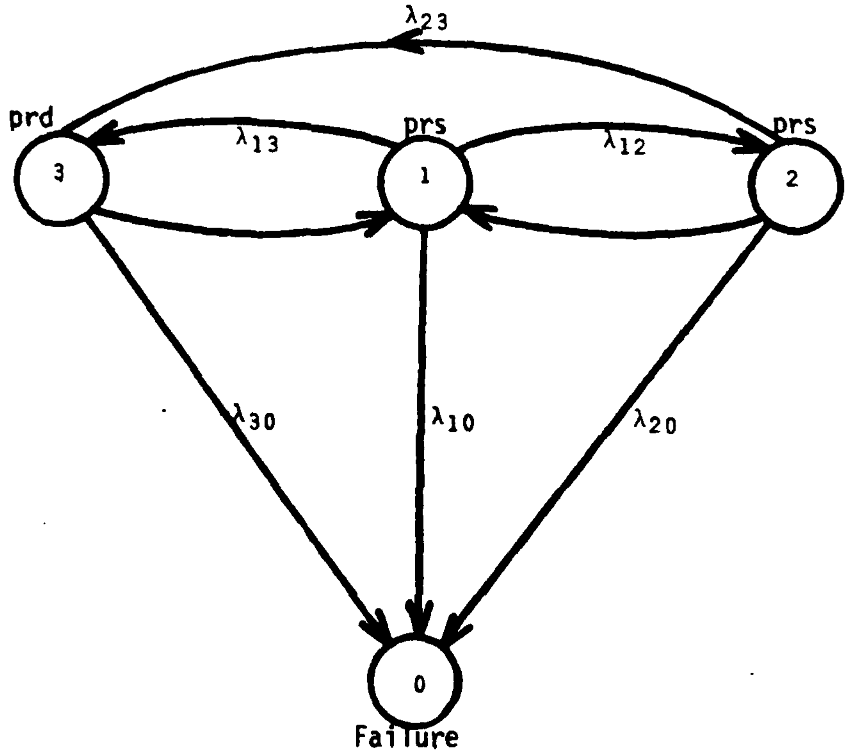

Figure 1. State Transition Diagram for a System with Two Types of Breakdowns and Repairs

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